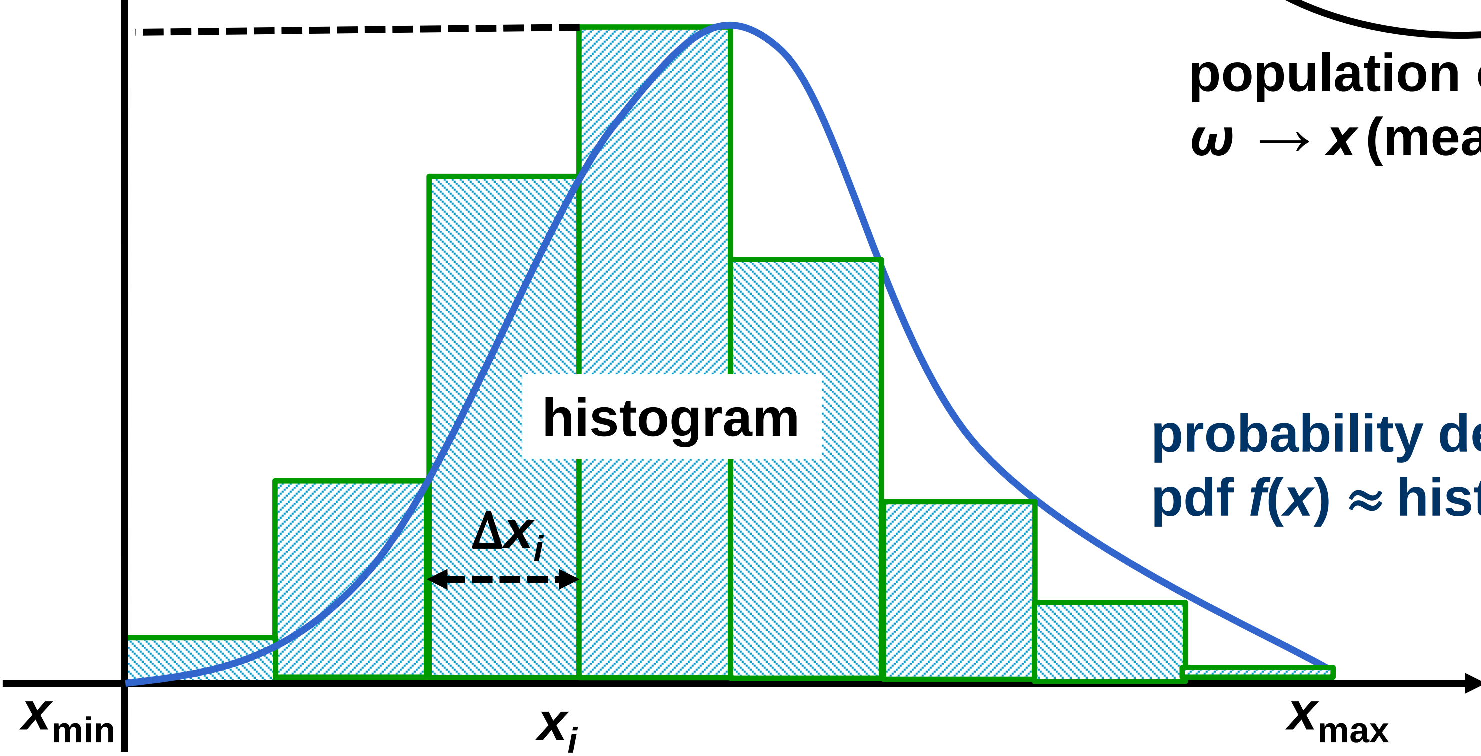
Random signals

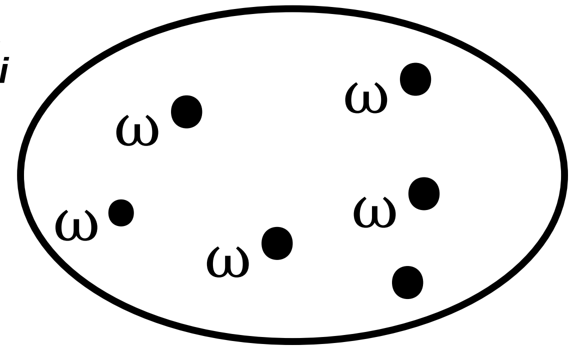
- Random variable
- Moments of a random variable
- Random signals
- Moments of a random signal
- Ergodicity of a random signal
- Spectral analysis of a random signal
- Summary
- Case studies

Random variable – a concept

n_i – # of measurements in an interval Δx_i
N – # of all measurements

$n_i/N = \Pr\{x \in \Delta x_i\}$





population of objects ω
$\omega \rightarrow x$ (measurable feature)

probability density function
pdf $f(x) \approx$ histogram

Probability density function (pdf) $f(x)$ is a closed form approximation (statistically verified) of an observable histogram.

Random variable

Random variable ***x*(*ω*) is a measurable feature changing in an unpredictable way in a population of objects *ω*; fluctuations are described by a cumulative distribution function *cpdf* (or probability density function - *pdf*).**

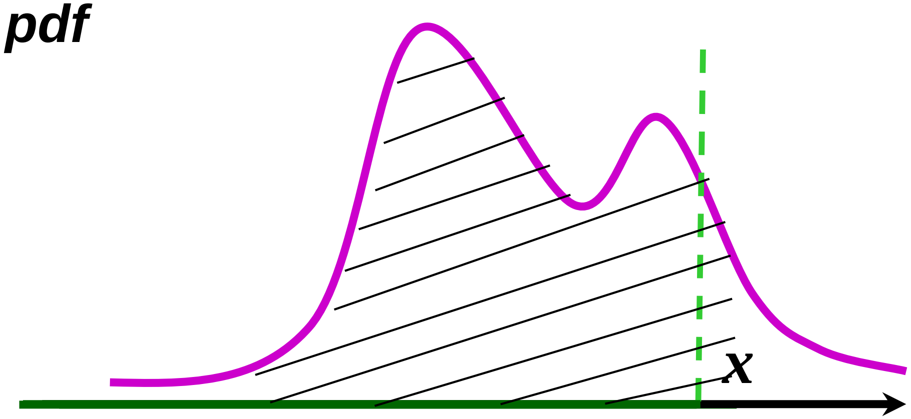
cumulative probability distribution function - *cpdf*

$$F(x) = \Pr\{\mathbf{x} \le x\}$$

probability density function - *pdf*

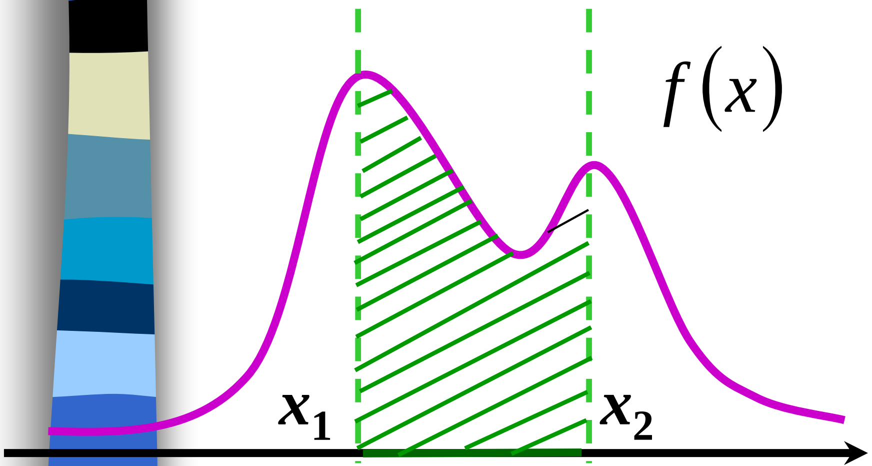
$$f(x) = dF(x)/dx$$

$$F(x) = \int_{-\infty}^{x} f(u)du$$



$$\Pr\{-\infty \le \mathbf{x} \le x\} = \int_{-\infty}^{x} f(u)du = F(x)$$

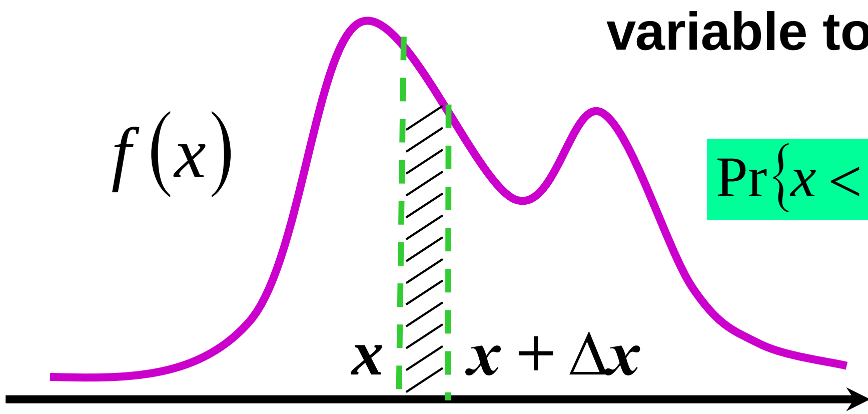
Probability Density Function (pdf) of a random variable



Pdf points to either:
- preferred values of a random variable,
- dispersion of a random variable around preferred values.

$$\Pr\{x_1 \le \mathbf{x} \le x_2\} = \int_{x_1}^{x_2} f(x)dx = F(x_2) - F(x_1)$$

Interval probability for a random variable to take on a specific value



$$\Pr\{x < \mathbf{x} \le x + \Delta x\} \approx f(x)\Delta x$$

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Normal random variable

$$\mathbf{N}(\mu,\sigma):\quad f(x)=\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-(x-\mu)^2/2\sigma^2\right]$$

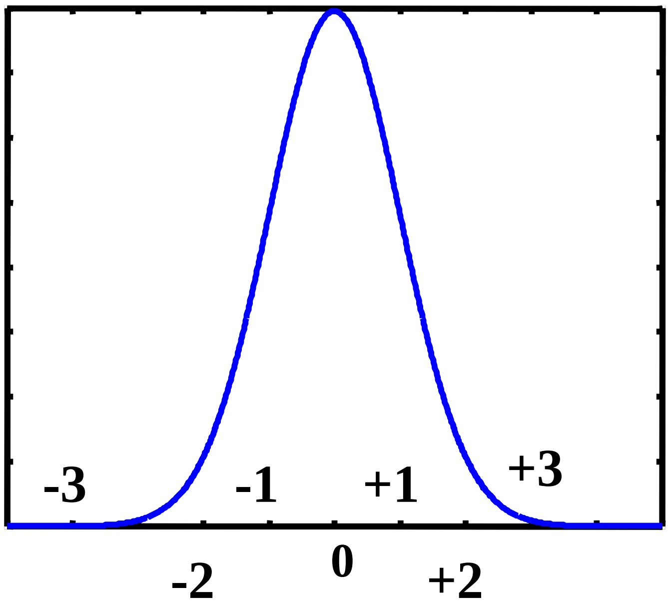
$$\mathbf{N}(0,1):\quad f(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)$$

$$\Pr\{|\mathbf{N}(0,1)|\le 1\}\approx 0{,}68$$

$$\Pr\{|\mathbf{N}(0,1)|\le 2\}\approx 0{,}95$$

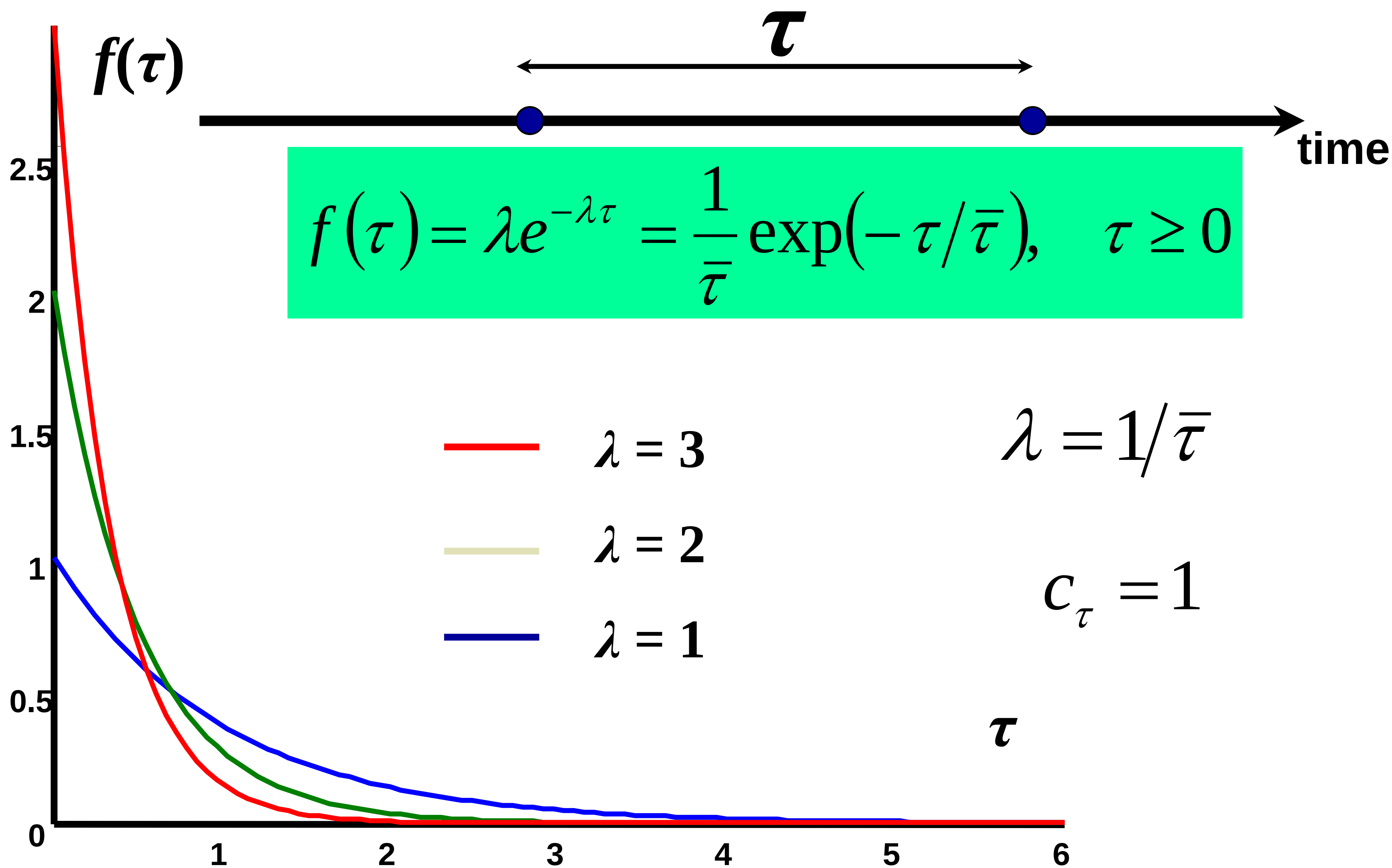
$$\Pr\{|\mathbf{N}(0,1)|\le 3\}\approx 0{,}997$$

$$\Pr\{|\mathbf{N}(0,1)|\le 4\}\approx 0{,}9999$$



Application: modeling Additive White Gaussian Noise (AWGN) corrupting transmission in analog and digital transmission channels.

Exponential random variable



$$f(\tau) = \lambda e^{-\lambda\tau} = \frac{1}{\bar{\tau}}\exp(-\tau/\bar{\tau}), \quad \tau \geq 0$$

$$\lambda = 1/\bar{\tau}$$

$$c_\tau = 1$$

Application: modeling call (session) interarrival times and session holding times.

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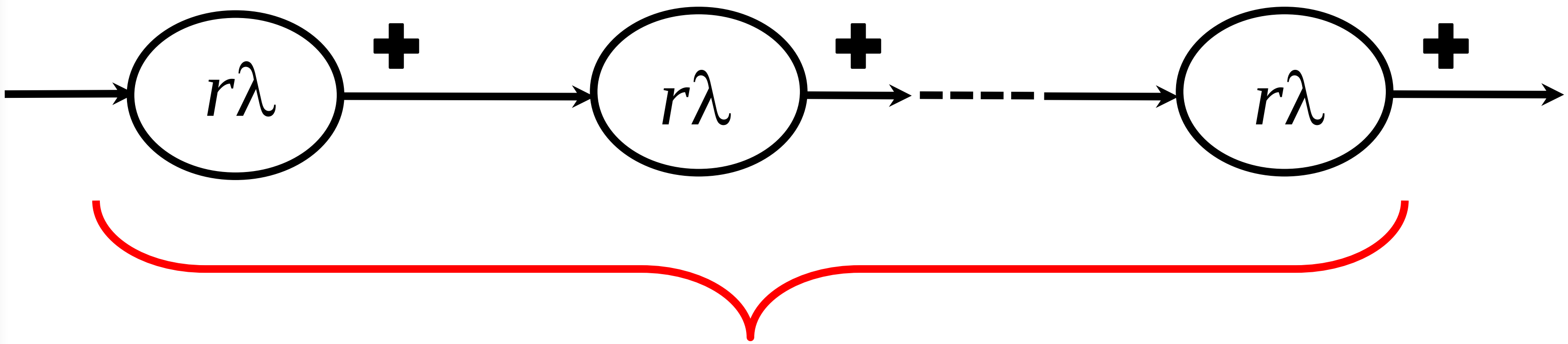
Erlang random variable



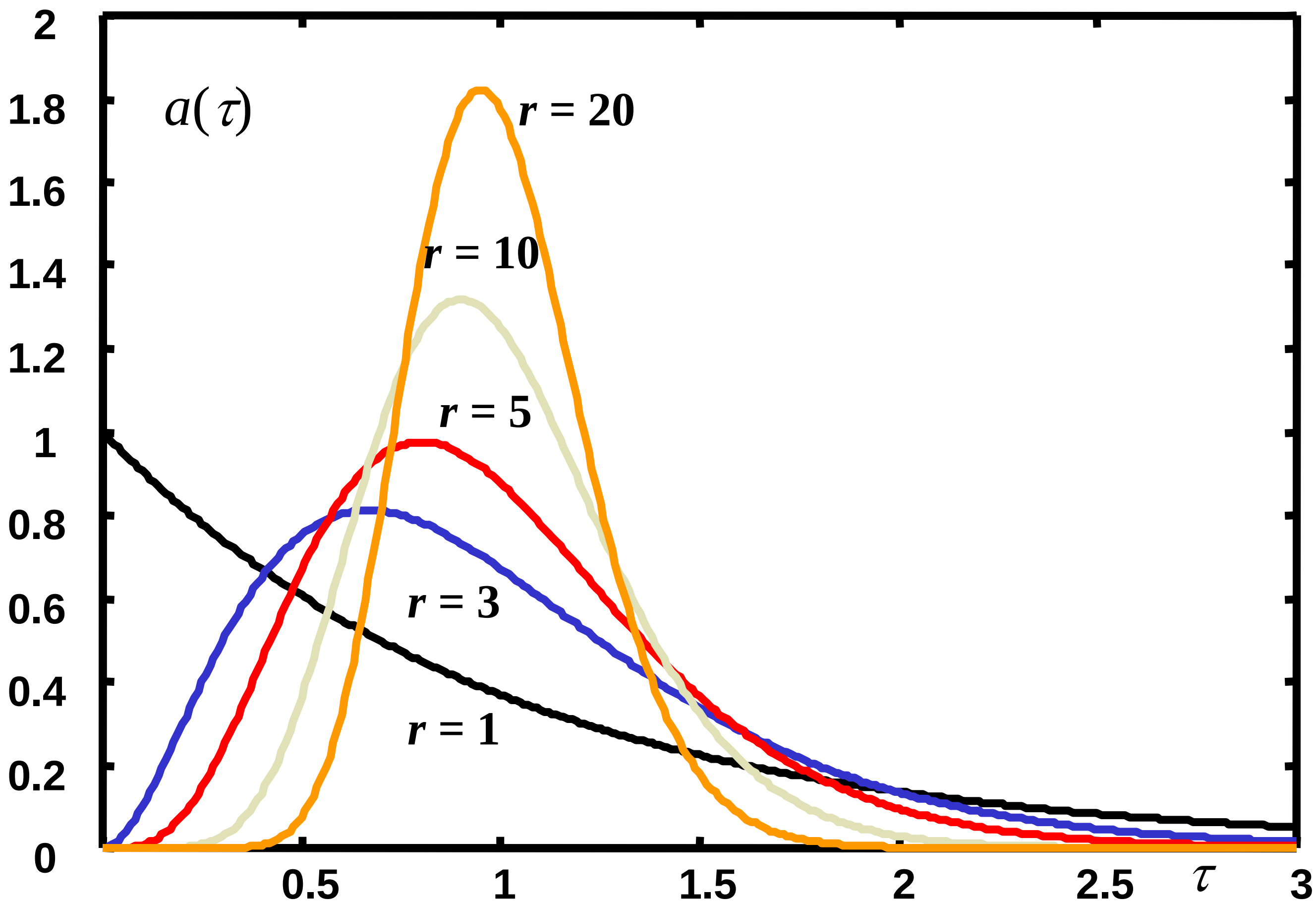
exponential random variables (#*r*)

$$f(\tau)=\frac{r\lambda(r\lambda\tau)^{r-1}e^{-r\lambda\tau}}{(r-1)!}$$

$$\boldsymbol{c}_{\tau}=\frac{1}{\sqrt{\boldsymbol{r}}}\le 1$$

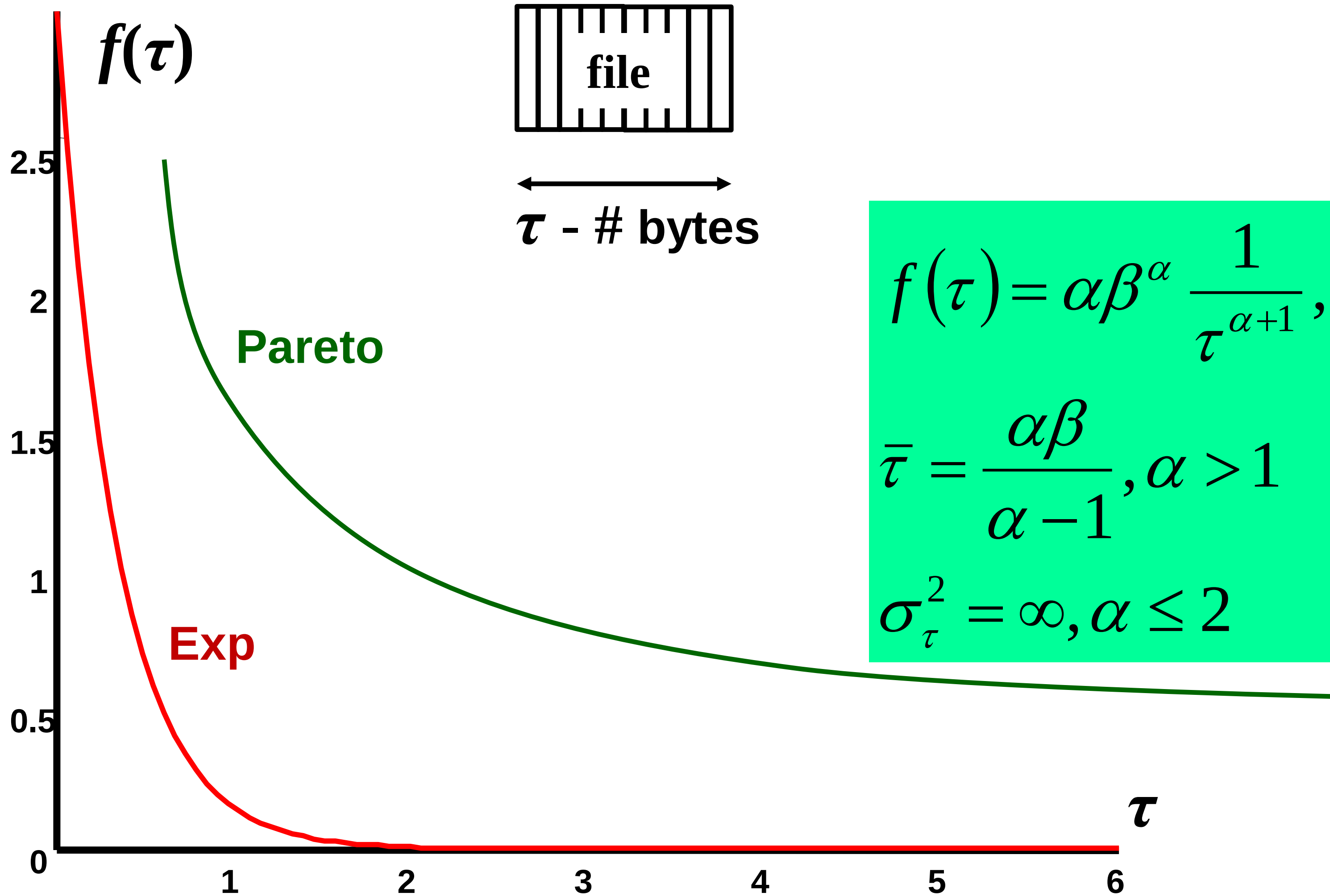
Application: modeling performance of central offices, switches, multiplexers, and concentrators.

Erlang random variable



Application: modeling performance of central offices, switches, multiplexers, and concentrators.

Pareto random variable



$$f(\tau) = \alpha\beta^{\alpha}\,\frac{1}{\tau^{\alpha+1}}, \tau \geq \beta$$

$$\bar{\tau} = \frac{\alpha\beta}{\alpha-1}, \alpha > 1$$

$$\sigma_{\tau}^{2} = \infty, \alpha \leq 2$$

Application: modeling long lasting files and sessions

Moments of a random variable

Mean (expected) value of a random variable:

$$\mathrm{E}\{\mathbf{x}\} = \overline{\mathbf{x}} = \int_{-\infty}^{\infty} x f(x)\,dx$$

Mean squared value of a random variable:

$$\mathrm{E}\{\mathbf{x}^2\} = \overline{\mathbf{x}^2} = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

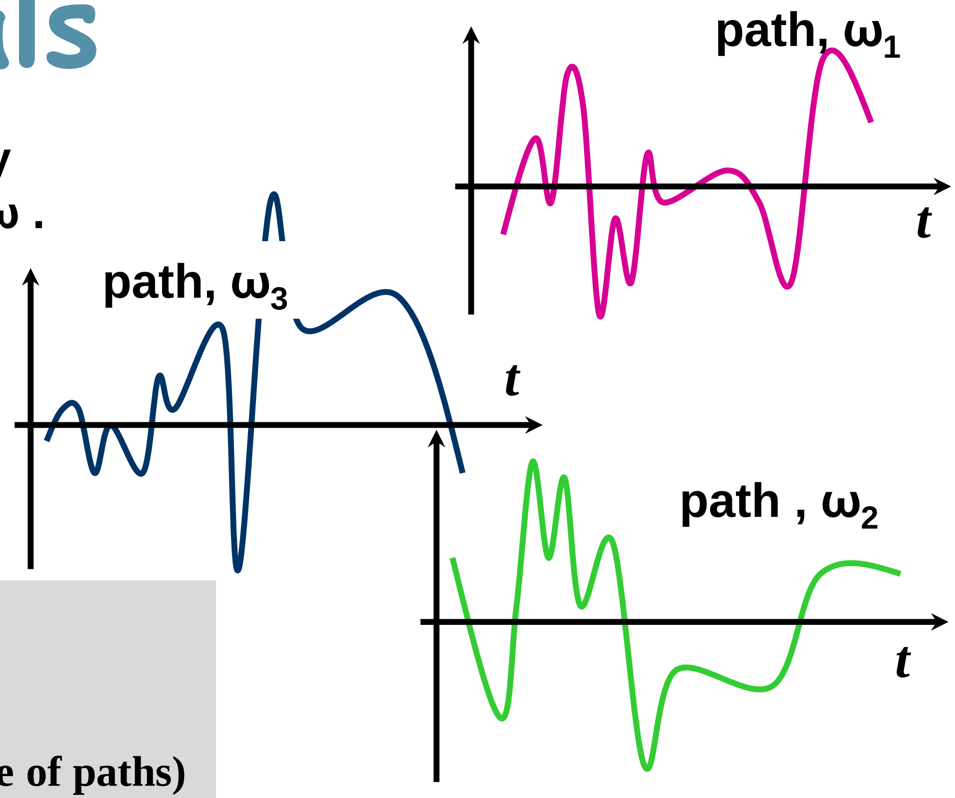
Any moment of a random variable is an <u>ensemble (population) average</u> as it is calculated over all objects (values) of the ensemble.

Random signals

Randomness **is not related** to a seemingly chaotic *path* of a single observed object ω .

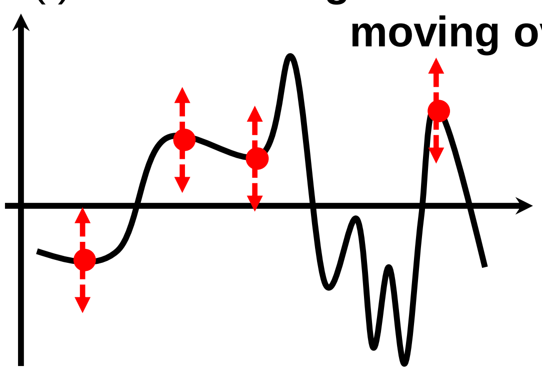
Path observed for an object ω is a **deterministic signal** (power?, energy?)

Randomness is grasped as an **ensemble of *paths*** of several objects $\omega_1, \omega_2, \omega_3 \ldots$

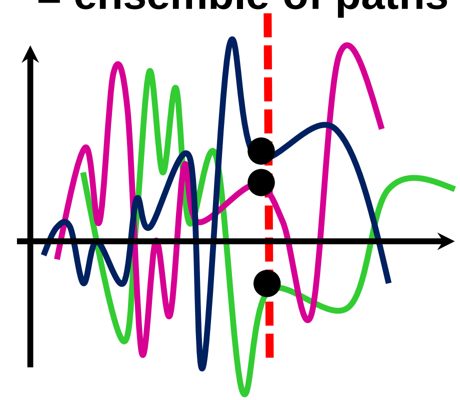


$\mathbf{x}(t=\text{const}, \omega=\text{const})$	– **sample (number)**
$\mathbf{x}(t=\text{const}, \omega=\text{var})$	– **random variable**
$\mathbf{x}(t=\text{var}, \omega=\text{const})$	– **path of a random signal**
$\mathbf{x}(t=\text{var}, \omega=\text{var})$	– **random signal (ensemble of paths)**

$\mathbf{x}(t)$ – random signal = random variable moving over time axis



$\mathbf{x}(t)$ - random signal = = ensemble of paths



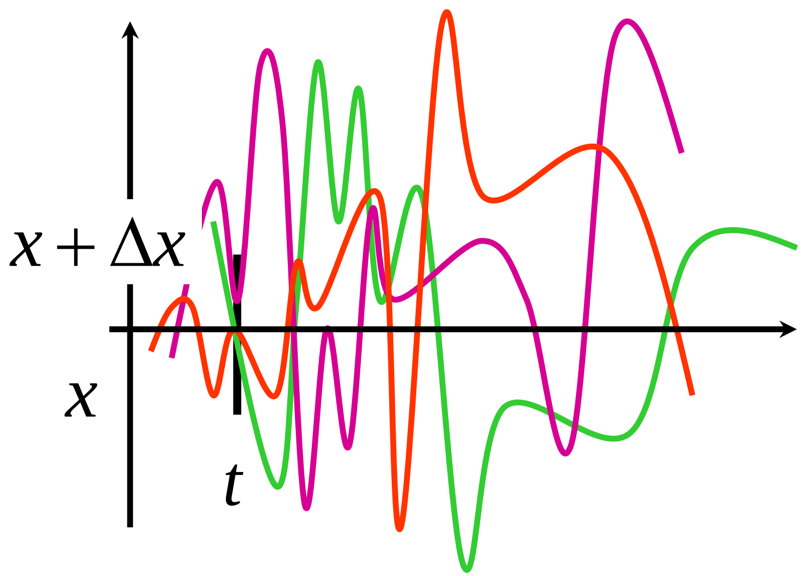
Practically, we adopt simpler notation $\mathbf{x}(\boldsymbol{t}) \equiv x(t)$.

Pdfs of random signals (stationary)

Probabilistic properties of a random signal <u>do not change with time</u>.

$$\Pr\{x < \mathbf{x}(t) \le x + \Delta x\} \approx f(x)\Delta x$$

Probability density function (of the 1st order)

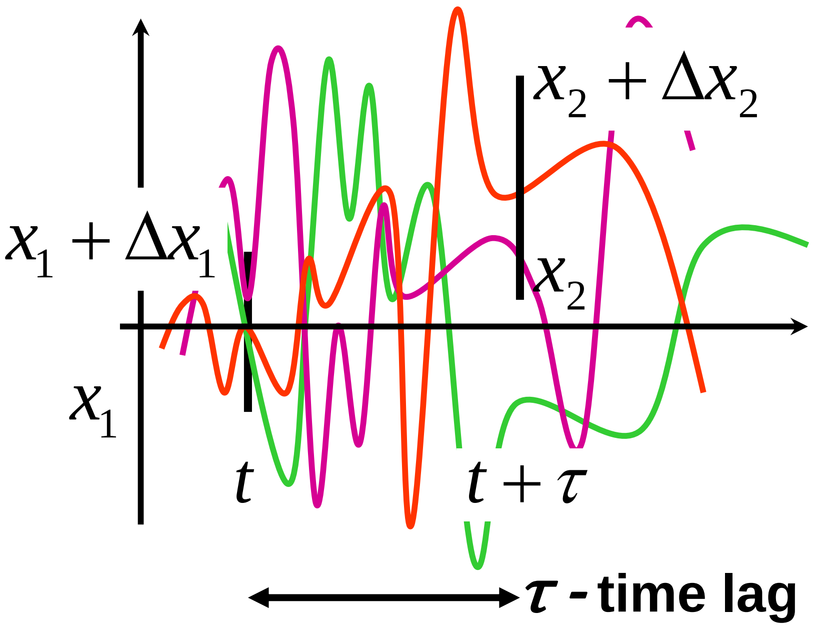


$f(x)\Delta x$ – **probability of the ensemble of paths „passing the gate" $[x,\ x+\Delta x]$ does not depend on a current time.**

Pdfs of random signals (stationary)

Pdf of the 2nd order does not depend on a current time though <u>depends on a time lag</u> $\boldsymbol{\tau}$)

$$\Pr\{(x_1 < \mathbf{x}(t) \le x_1 + \Delta x_1) \wedge (x_2 < \mathbf{x}(t+\tau) \le x_2 + \Delta x_2)\} \approx$$
$$\approx f(x_1, x_2; \tau)\Delta x_1 \Delta x_2$$



Pdfs of both 1st and 2nd order do not depend on a current time (stationary random signal).

$\boldsymbol{f(x_1,x_2;\tau)\Delta x_1\Delta x_2}$ **– probability of the ensemble of paths „passing both gates” $[x_1, x_1+\Delta x_1]$ and $[x_2, x_2+\Delta x_2]$ over time lag τ.**

Moments of a random signal

$$\mathrm{E}\{\mathbf{x}(t)\} = \bar{x} = \int_{-\infty}^{\infty} x f(x) dx = const$$

$$\mathrm{E}\{\mathbf{x}^2(t)\} = \overline{x^2} = \int_{-\infty}^{\infty} x^2 f(x) dx = const$$

The mean values of a stationary random signal <u>do not depend</u> on a current time.

Mean values of a random proces are ensemeble averages as they are calculated over all process paths.

Moments of a random signal

Autocorrelation function (acf) generalizes the 2nd order moment to two time instants delayed by τ. Acf does not depend on a current time, yet it depends on a delay τ.

$$\overline{R}(\tau) = \mathrm{E}\{x(t)x(t+\tau)\} = \overline{x(t)x(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau)\, dx_1 dx_2$$

Autocorrelation function is a measure of proces correlation at two different time instants delayed by τ.

$$0 \le \left|\overline{R}(\tau)\right| \le 1 \quad \left(\text{for } \overline{x} = 0 \text{ and } \overline{x^2} = 1\right)$$

$$\left|\overline{R}(\tau)\right| = 0 - \textbf{no correlation} \qquad \left|\overline{R}(\tau)\right| = 0 - \textbf{full correlation}$$

Fourier transform of the autocorrelation function provides the power spectrum of a random proces (Wiener-Chinczyn theorem)

$$\mathcal{F}\{\overline{R}(\tau)\} = \overline{S}(\omega)$$

Two special cases of interest

$$\mathrm{E}\{x(t)x(t+\tau)\} = \overline{x(t)x(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) dx_1 dx_2$$

Case #1 – $\tau = 0$

$$\mathrm{E}\{x(t)x(t+\tau)\}\big|_{\tau=0} = \overline{x(t)x(t)} = \overline{x^2} = \int_{-\infty}^{\infty} x^2 f(x) dx$$

Case #2 – $x(t)$ and $x(t+\tau)$ are independent

$$\mathrm{E}\{x(t)x(t+\tau)\} = \overline{x(t)x(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1;\tau) f(x_2;\tau) dx_1 dx_2 =$$

$$= \int_{-\infty}^{\infty} x_1 f(x_1;\tau) dx_1 \int_{-\infty}^{\infty} x_2 f(x_2;\tau) dx_1 = (\overline{x})^2$$

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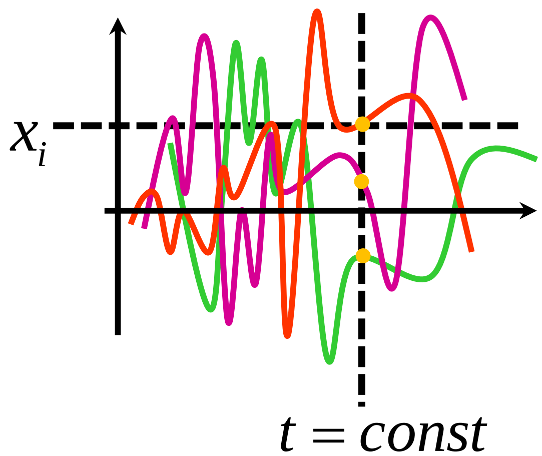
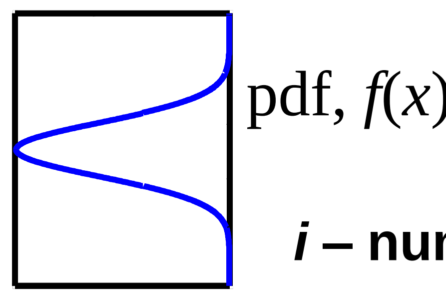
Ergodocity of a random signal

Moments of a random signal are *ensemble* (*probabilistic*) *averages* as they are calculated for a fixed time instants for an <u>ensemble including all signal paths</u>.



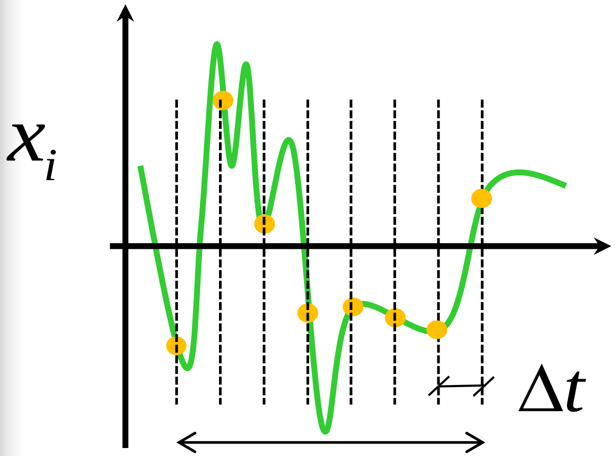


i **– number of a path**
n$_i$ **– number of *x$_i$* samples**
N **- # all paths**

$$\frac{\sum_i n_i x_i}{N} = \sum_i x_i \frac{n_i}{N} = \sum_i x_i \Pr\{x_i\} \approx \int_{-\infty}^{\infty} x f(x)\,dx = \underset{\substack{\text{ensemble}\\ \text{average}}}{\bar{x}}$$

Ergodicity of a random signal

Time-averaged **moments of a signal are determined for a single signal path.**



$$T = N\Delta t$$

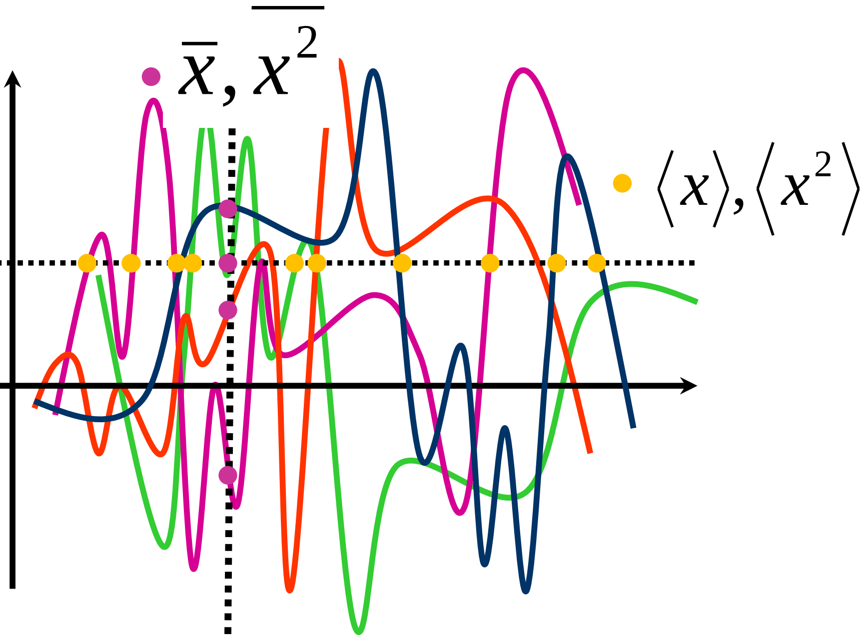
$$1/N = \Delta t/T$$

i **– sample number**
N **– # all samples**

$$\frac{\sum_i x_i}{N} = \frac{1}{T}\sum_i x_i \Delta t = \underbrace{\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,dt}_{\text{time average}} = \langle x \rangle$$

Ergodicity of a random signal

Ergodic random signal features that a single signal path represents an ensemble of all signal paths.



$$\underset{\text{ensemble average}}{\bar{x}} \overset{?}{=} \underset{\text{time average}}{\langle x \rangle}$$

ergodicity

$$\underset{\text{ensemble average}}{\overline{x^2}} \overset{?}{=} \underset{\text{time average}}{\langle x^2 \rangle}$$

ergodicity

Definition of an ergodic random signal means that:

- **⟨time-averaged moments⟩ of a random signal determined for each single signal path are equal to**
- **$\overline{\text{ensemble (probabilistic) moments}}$ calculated for an entire ensemble of signal paths.**

Ergodicity of a mean squared value

single path

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}x^2(t)dt=\left\langle x^2\right\rangle<\infty$$

time average

random process

$$\underset{\textit{ensemble average}}{\overline{x^2}}=\mathrm{E}\left\{x^2\right\}=\int_{-\infty}^{\infty}x^2 f(x)dx<\infty$$

Single path is a power signal

$\langle x^2\rangle<\infty$

$$\left\langle x^2\right\rangle=\overline{x^2}$$

Ergodicity

1. Paths of a random ergodic signal are power signals (unperiodic) so they do not possess a Fourier transform.

2. For each path of a random ergodic signal (power signal) it is possible to find its autocorrelation function (averaging over time) and then its power spectral density via the Fourier transform.

3. Power spectrum density of a random signal can be obtained by ensemble averaging of power density spectra of signal paths.

Wiener-Chinczyn theorem

Spectral analysis of a random signal

<u>Wiener – Chinczyn theorem</u>
Power spectrum of a stationary random signal is a Fourier transform of its autocorrelation function.

$$\overline{R}(\tau) \xleftrightarrow{\mathcal{F}} \overline{S}(\omega)$$

$$\overline{R}(\tau) = \overline{x(t)x(t+\tau)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2; \tau) dx_1 dx_2$$

The autocorrelation is to be calculated in the probability domain (pdf of the 2nd order) instead in the time domain.
This property is important as random processes are described via probability density functions (of the 1st and 2nd order).

Note that:

- **pdf of the 1st order describes dispersion of random signal values**
- **pdf of the 2nd order (autocorrelation function) describes power density spectrum of a random signal (after Fourier transformation)**

Spectral analysis of a random signal

<u>Addendum to Wiener – Chinczyn theorem</u>

Power spectrum of a <u>cyclostationary random process</u> (its autocorrelation function is periodic in *t* with a period *T*) is a Fourier transform of its time-averaged autocorrelation function (over a period).

$$\overline{R}(t,\tau)=\overline{R}(t+T,\tau)$$

$$\left\langle \overline{R}(t,\tau)\right\rangle=\frac{1}{T}\int_{-T/2}^{+T/2}\overline{R}(t,\tau)dt$$

$$\left\langle \overline{R}(t,\tau)\right\rangle\xleftrightarrow{\mathcal{F}}\overline{S}(\omega)$$

Wiener-Chinczyn theorem – a proof

1. Paths of a random ergodic signal are <u>power signals</u> (unperiodic) so they do not possess a Fourier transform.

2. <u>For each path</u> of a random ergodic signal (power signal) it is possible to find its <u>autocorrelation function</u> (averaging over time) and then its power spectral density via the Fourier transform.

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}x(t)x(t+\tau)dt=\langle x(t)x(t+\tau)\rangle=\langle R(\tau)\rangle\xrightarrow{\mathcal{F}}\langle S(\omega)\rangle=\mathcal{F}\{\langle R(\tau)\rangle\}$$

3. <u>Power spectrum density of a random signal</u> can be obtained by <u>ensemble averaging of power density spectra of signal paths</u>.

$$\bar{S}(\omega)=\mathrm{E}\{\langle S(\omega)\rangle\}=\mathrm{E}\{\mathcal{F}\{\langle R(\tau)\rangle\}\}=\mathcal{F}\{\mathrm{E}\{\langle R(\tau)\rangle\}\}=$$

$$=\mathcal{F}\left\{\mathrm{E}\left\{\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}x(t)x(t+\tau)dt\right\}\right\}=\mathcal{F}\left\{\left\{\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}\mathrm{E}[x(t)x(t+\tau)t]\right\}dt\right\}=$$

$$=\mathcal{F}\left\{\left\{\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2}\bar{R}(\tau)dt\right\}\right\}=\mathcal{F}\{\bar{R}(\tau)\}=\bar{S}(\omega)$$

Summary

- Random signal exhibits chaotic fluctuations described by a random variable.
- Random signal is an ensemble of deterministic paths which are power signals.
- Time averages and ensemble averages are defined for random signals. Ensemble averages do not depend on current time for stationary random signals.
- Time averages are equal to enesemble averages for ergodic processes.
- Power spectrum of an ergodic random signal is a Fourier transform of its autocorrelation function (Wiener-Chinczyn theorem).

Case study – amplitude modulation

$x(t)$ – stationary stochastic process, $R_y(\tau)$

$y(t) = x(t)\cos\omega_0 t$ – amplitude modulation

$$R_y(t,\tau) = E\{x(t+\tau)\cos\omega_0(t+\tau)x(t)\cos\omega_0 t\}$$

$$R_y(t,\tau) = E\{x(t+\tau)x(t)\}\cos\omega_0(t+\tau)\cos\omega_0 t$$

$$R_y(t,\tau) = R_x(\tau)\cos\omega_0(t+\tau)\cos\omega_0 t$$

$$R_y(t,\tau) = \frac{1}{2}R_x(\tau)\cos\omega_0\tau + \frac{1}{2}R_x(\tau)\cos(2\omega_0 t+\tau)$$

$$R_y(\tau) = \langle R_y(t,\tau)\rangle = \frac{1}{2}R_y(\tau)\cos\omega_0\tau$$

$$P_y = R_y(0) = \frac{1}{2}R_x(0) = \frac{1}{2}\overline{x^2}$$

$$S_y(\omega) = \frac{1}{4}\left[S_x(\omega-\omega_0)+S_x(\omega+\omega_0)\right]$$

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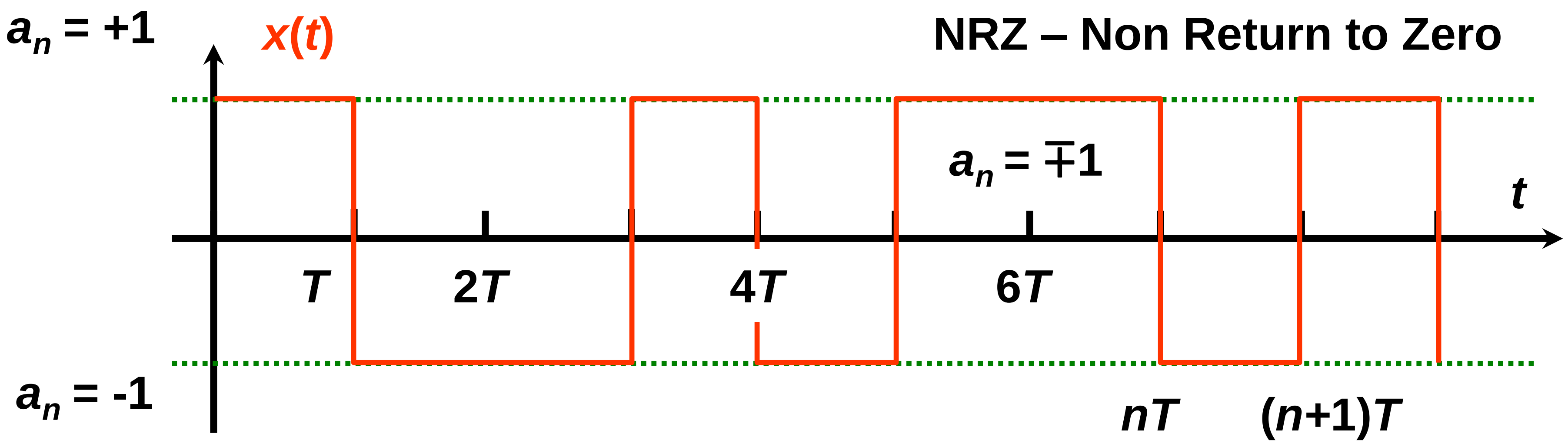
Case study – bipolar NRZ line code



$$\Pr\{a_n = +1\} = p$$

$$\Pr\{a_n = -1\} = q = 1 - p \qquad \mathrm{E}\{a_n\} = +1 \times p + (-1) \times q = p - q$$

$$\mathrm{E}\{a_k a_n\} = \begin{cases} \mathrm{E}\{a_n^2\} = 1, k = n \\ \mathrm{E}\{a_k\} E\{a_n\} = (p-q)^2 = \sigma^2, k \neq n \end{cases}$$

Symbols a_n are iid random variables.
(iid = independent identically distributed)

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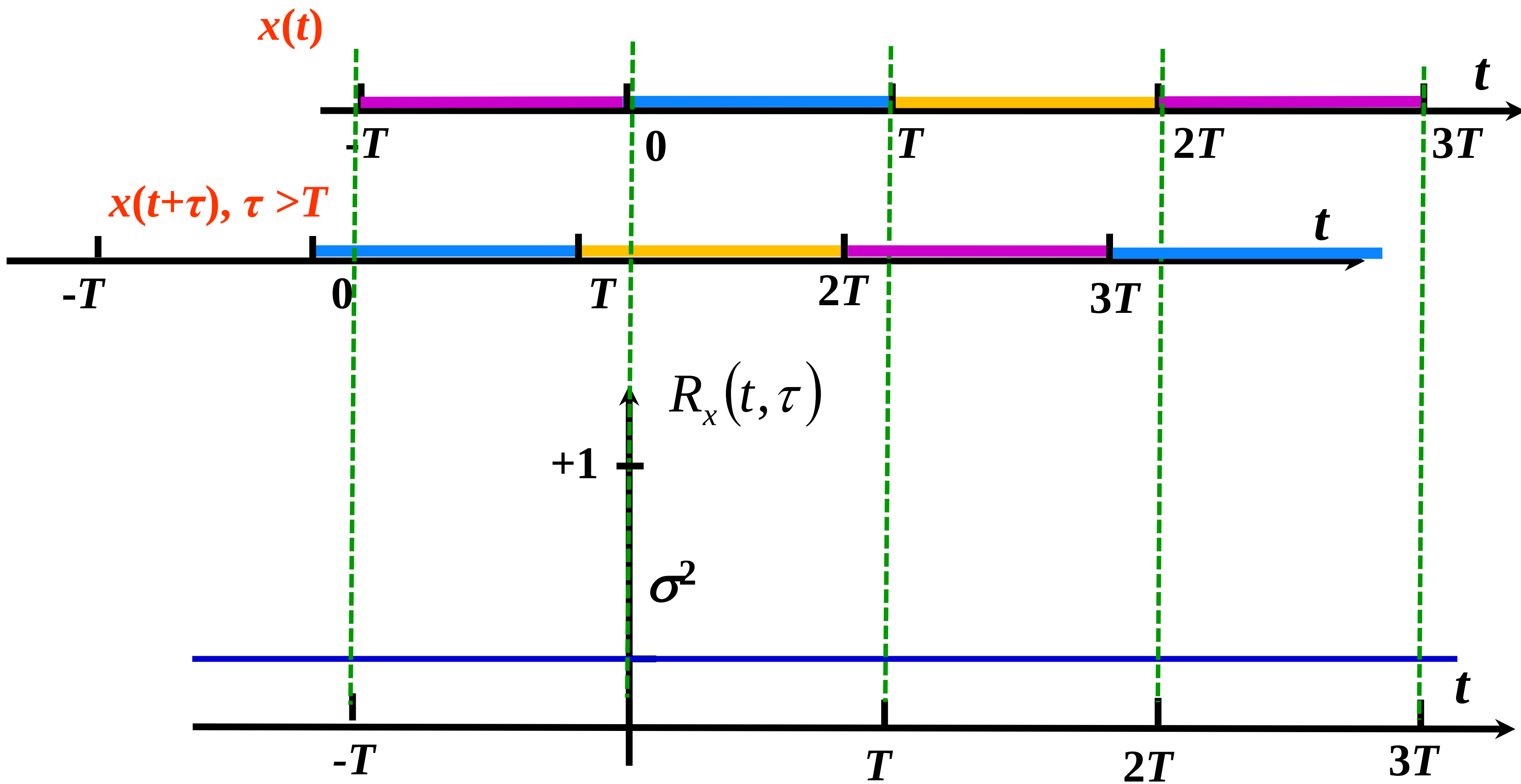
Bipolar NRZ line code - ACF

$|\tau| \geq T$

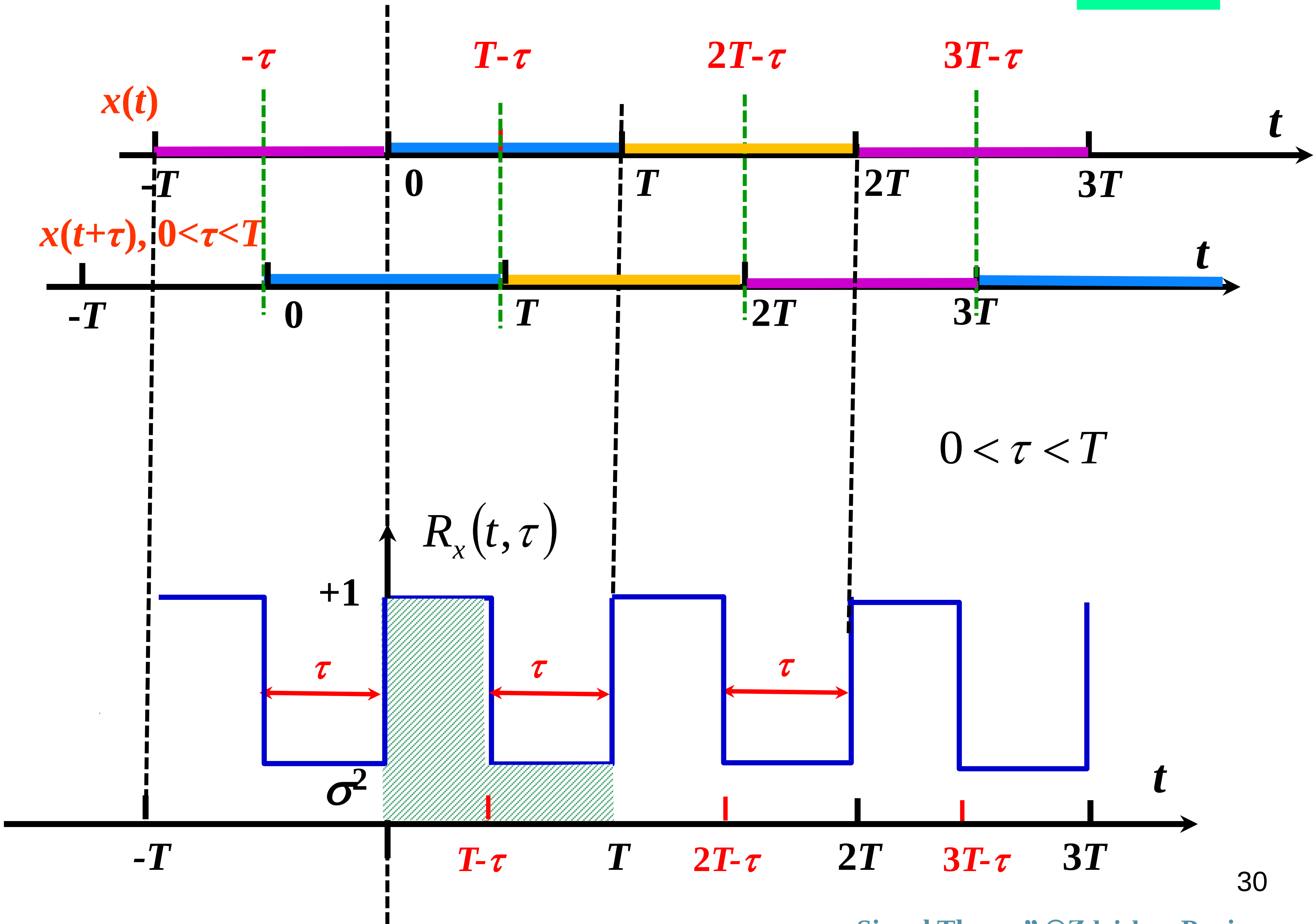


$$R_x(\tau) = \langle R_x(t,\tau) \rangle = \sigma^2$$

$$|\tau| \geq T$$

Bipolar NRZ line code - ACF

$|\tau| \le T$



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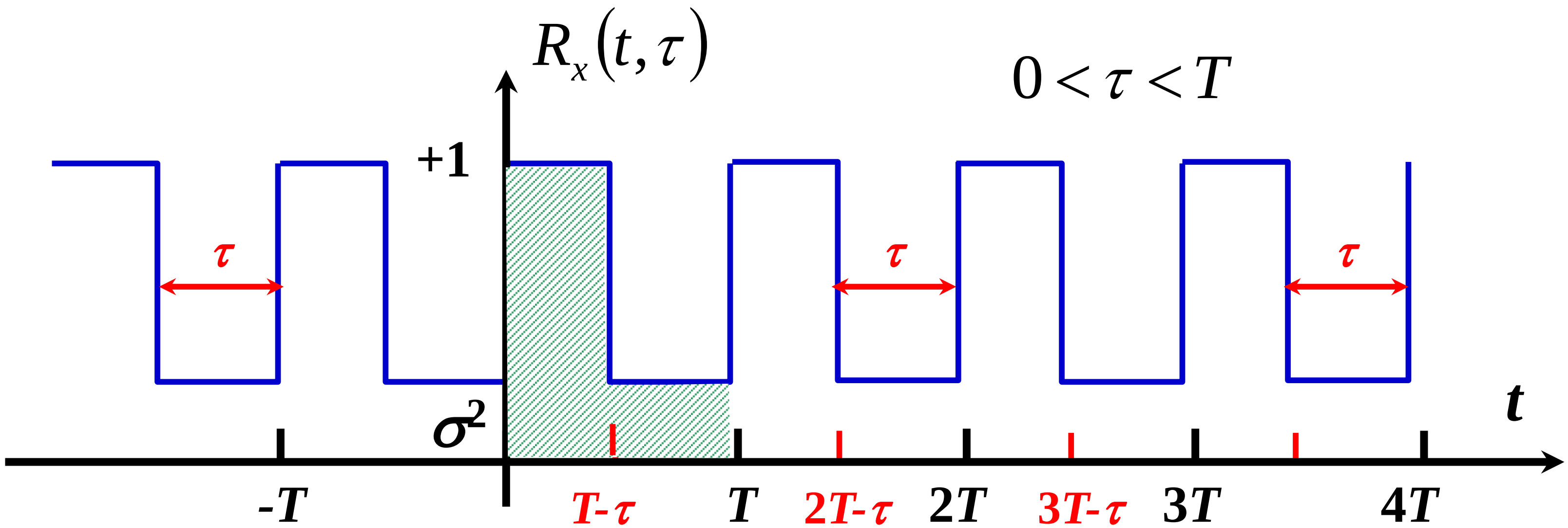
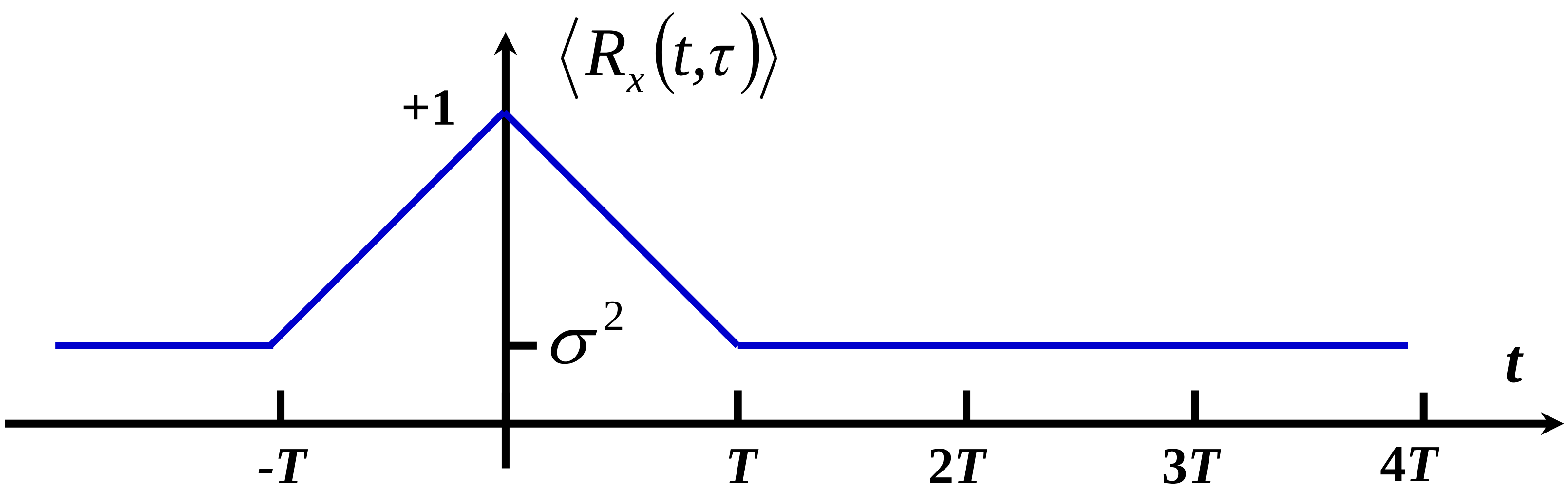
Bipolar NRZ line code - ACF $|\tau| \le T$



$$\langle R_x(t,\tau)\rangle = \frac{1\times(T-\tau)+\sigma^2\times\tau}{T} = 1-\left(1-\sigma^2\right)\frac{|\tau|}{T}$$

$$|\tau| \le T$$

Bipolar NRZ line code – ACF



$$\langle R_x(t,\tau)\rangle = \sigma^2 + (1-\sigma^2)\Lambda_{2T}(\tau)$$

$$\overline{S_x}(\omega) = \mathcal{F}\{\langle R_x(\tau)\rangle\} = 2\pi\sigma^2\delta(\omega) + (1-\sigma^2)T\mathrm{Sa}^2\frac{\omega T}{2}$$

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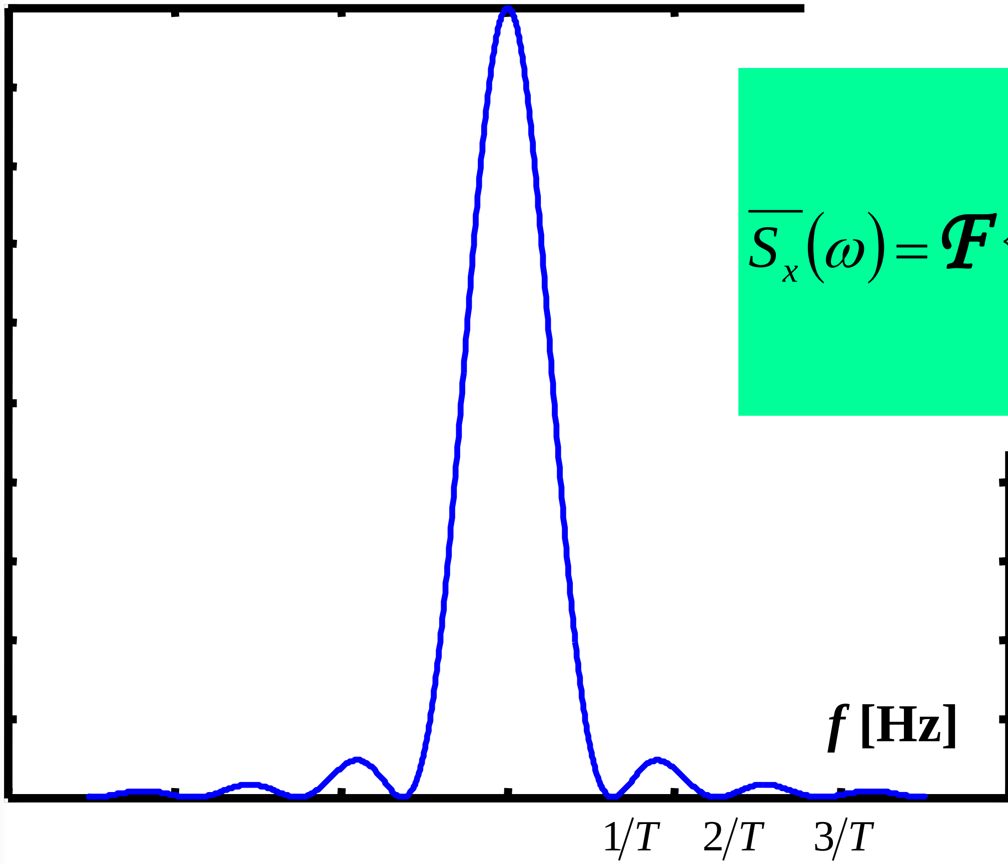
Bipolar NRZ line code – power spectrum



$$\overline{R_x}(\tau)=\Lambda_{2T}(\tau)$$

$$\overline{S_x}(\omega)=\mathcal{F}\left\{\overline{R_x}(\tau)\right\}=T\mathrm{Sa}^2\frac{\omega T}{2}=T\mathrm{Sa}^2\pi fT$$

$$p=q=1/2$$

Power spectrum is concentrated in the bandwidth 0 < *f* < 1/*T*; Nyquist theorem (ISI = 0) states that the half of the spectrum 0 < *f* < 1/(2*T*) is sufficient (provided synchronous resampling).

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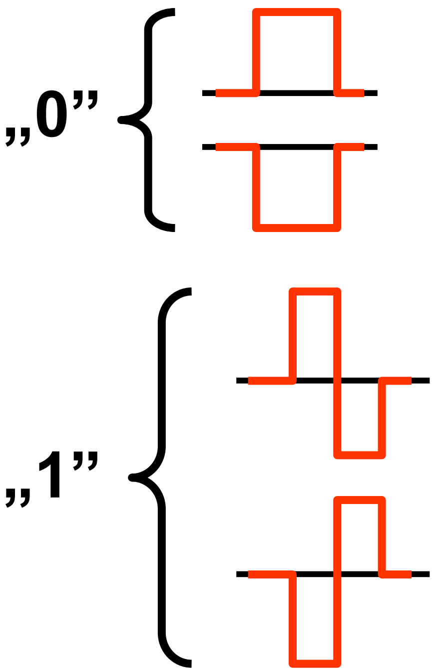
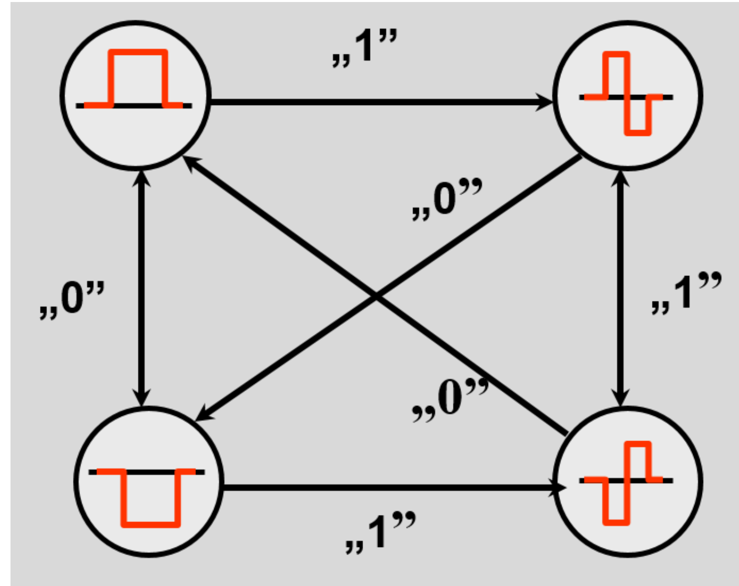
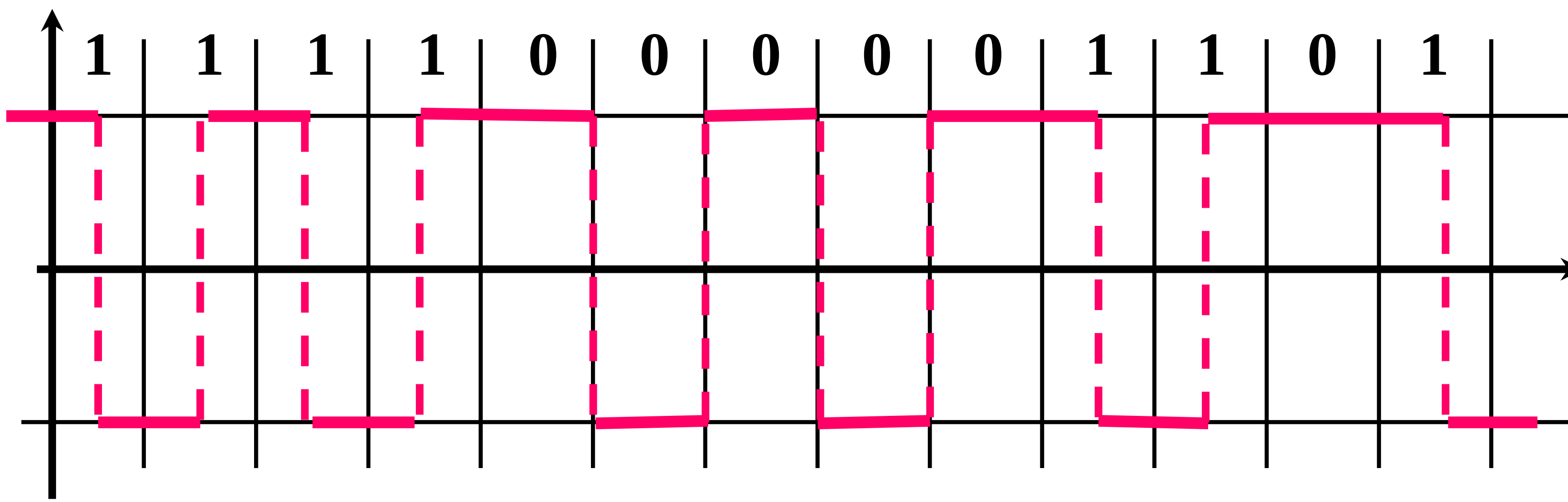
Miller’s line code



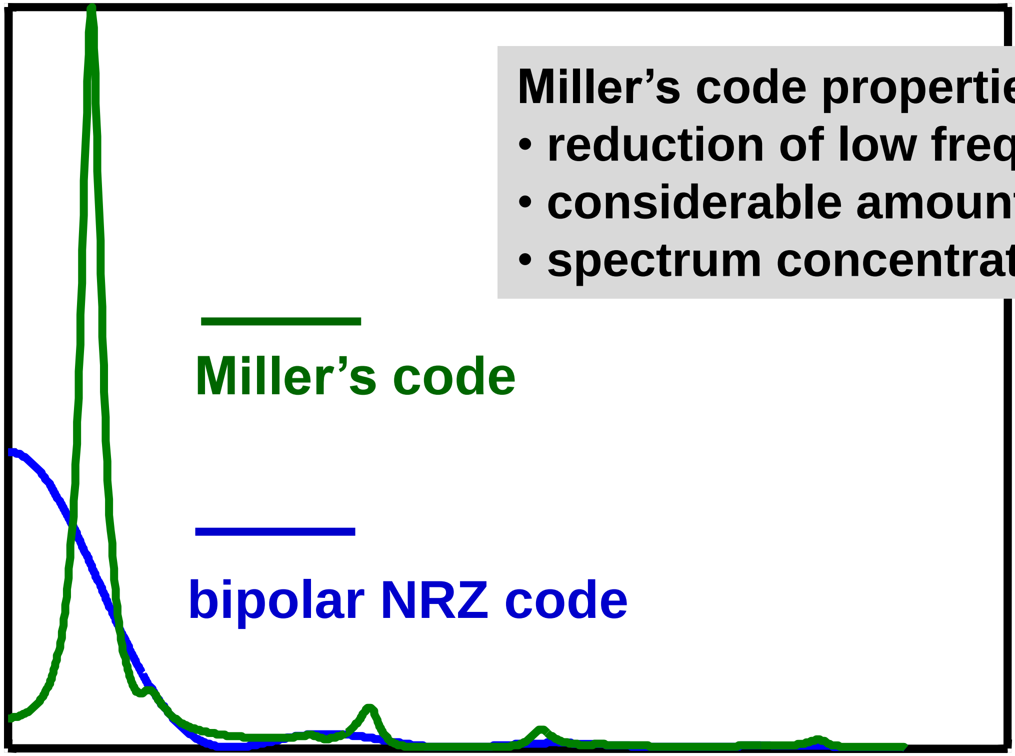
• keeping signal level at transitions „1” → „0”
• opposite signal level at transistions „0” → „0”

• maintaining signal level at transitions „1” → „1”
• maintaining signal level at transitions „0” → „1”





Miller’s line code – power spectrum



Miller’s code properties:
- reduction of low frequency spectrum components
- considerable amount of timing content
- spectrum concentrated in a relatively narrow bandwidth

$$\overline{S_x}(\omega) = \frac{23 - 2\cos\pi fT - 22\cos 2\pi fT - 12\cos 3\pi fT + 5\cos 4\pi fT}{2\pi^2 f^2 T(17 + 8\cos 8\pi fT)} +$$

$$+ \frac{12\cos 5\pi fT + 2\cos 6\pi fT - 8\cos 7\pi fT + 2\cos 8\pi fT}{2\pi^2 f^2 T(17 + 8\cos 8\pi fT)}$$

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